

Sequences and Series

ISI & CMI Yearlong Course 2024

★ Sequences

⊙ Introduction

- Suppose A is countable, S is a set. $f: A \rightarrow S$.
↳ can assume $A = \mathbb{N}$. $\nrightarrow$ if not, let ϕ be the bijection $\mathbb{N} \rightarrow A$. Consider $f \circ \phi$.

Denote $f(1) =: s_1, f(2) =: s_2, \dots$

Thus we have a sequence $s_1, s_2, \dots$ written as: $\{s_n\}_{n \in A} \subseteq S \rightarrow$ "space" set. (where are the elements.)
↳ indexing set (for us, countable)

- We will mostly assume S to be $\mathbb{R}$.

• Convergence

Written as: $a_n \rightarrow a$ (as $n \rightarrow \infty$)
limit of $\{a_n\}$

We say $\{a_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$ converges iff $\exists (a) \in \mathbb{R}$
There is some num.

$\forall \varepsilon > 0, \exists N \in \mathbb{N} \mid \forall n \geq N, n \in \mathbb{N}$
"arbitrarily" close after a certain point in the sequence.

$d(a_n, a) < \varepsilon$ $\rightarrow$ values get close enough.

where d is the Euclidean metric on $\mathbb{R}$

$$\hookrightarrow d(x, y) = |x - y|$$

- Example: $\{a_n = \frac{1}{n}\}_{n \in \mathbb{N}} \subseteq \mathbb{R}, \quad a_n \rightarrow a := 0$

Books: ① S. Ramarean ← easy
 ② Bartle-Shurbert ← default ✓
 ③ Rudin ← advanced } for this chapter.

• Monotonicity

Suppose $\{a_n\} \subseteq \mathbb{R} \mid \forall n \in \mathbb{N} \quad a_n \leq a_{n+1}$
 Then $\{a_n\}$ is said to be monotonically increasing.

Suppose $\{a_n\} \subseteq \mathbb{R} \mid \forall n \in \mathbb{N} \quad a_n < a_{n+1}$
 Then $\{a_n\}$ is said to be strictly increasing.

(Similarly for decreasing)

A sequence that is either monotonically increasing or monotonically decreasing is called "monotone".

↳ strict $\Rightarrow$ monotone.

• Boundedness

don't worry m.

{ A sequence $\{a_n\}$ in a metric space S is said to be bounded iff $\exists R > 0 \mid \forall n \in \mathbb{N} \quad a_n \in B(0, R)$.

If $S = \mathbb{R}$, $\{a_n\}$ is said to be...

i) ...bounded above: iff $\exists M \in \mathbb{R} \mid \forall n \in \mathbb{N} \quad a_n < M$

ii) ...bounded below: iff $\exists m \in \mathbb{R} \mid \forall n \in \mathbb{N} \quad a_n > m$

iii) ...bounded: iff it is both bounded above & bounded below

• Example $\{a_n := \frac{1}{n}\} \subseteq \mathbb{R}$

$\{a_n\}$ is strictly decreasing. $\{a_n\}$ is bounded below by 0.

- Lemma The limit of a convergent sequence is unique

Fix any $\varepsilon > 0$. Show show $|a - a'| < 2\varepsilon$.

We have, $\exists N_1 \in \mathbb{N} \mid \forall n > N_1, n \in \mathbb{N} \mid |a_n - a| < \varepsilon$
 $\exists N_2 \in \mathbb{N} \mid \forall n > N_2, n \in \mathbb{N} \mid |a_n - a'| < \varepsilon$.

Take $n = \max\{N_1, N_2\}$

$\therefore |a_n - a| < \varepsilon, |a_n - a'| < \varepsilon$

By triangle inequality, $|a - a'| = |a - a_n + a_n - a'| \leq |a - a_n| + |a_n - a'| < \varepsilon + \varepsilon < 2\varepsilon$ $\square$

- Monotone Convergence Theorem

Suppose $\{a_n\} \subseteq \mathbb{R}$ is monotonically increasing & bounded above.

Then: $a_n \rightarrow a := \sup \{a_n \mid n \in \mathbb{N}\}$.

(same for decreasing, bounded below, inf)

Proof:

Fix any $\varepsilon > 0$.

$\therefore a = \sup \{a_n\}$, $a - \varepsilon$ is not an upper bound of $\{a_n\}$

$\Rightarrow \exists N \in \mathbb{N} \mid a - \varepsilon < a_N \leq a$

Now, $\therefore \{a_n\}$ is monotonically increasing --

$\forall n > N, n \in \mathbb{N}, a - \varepsilon < a_N \leq a_n \leq a$

$\Rightarrow \forall n > N, n \in \mathbb{N}, a - a_n < \varepsilon$ $\square$

This completes the proof.

• Standard Arithmetic Results

$$\{a_n\} \in \mathbb{R}, \{b_n\} \in \mathbb{R}. \quad a_n \rightarrow a, \quad b_n \rightarrow b$$

$$i) \quad a_n + b_n \rightarrow a + b$$

$$ii) \quad \lambda a_n \rightarrow \lambda a \quad \forall \lambda \in \mathbb{R}$$

$$iii) \quad a_n b_n \rightarrow ab$$

$$iv) \quad \text{If } \forall n, \frac{a_n}{b_n} \rightarrow \frac{a}{b}.$$

Proof: (abuse of Δ inequality)

$$i) \quad |a_n + b_n - a - b| \leq |a_n - a| + |b_n - b|$$

$$\text{Fix any } \varepsilon > 0, \quad \begin{array}{l} \exists N_1 \in \mathbb{N} \\ \exists N_2 \in \mathbb{N} \end{array} \mid \begin{array}{l} \forall n \geq N_1, |a_n - a| < \varepsilon/2 \\ \forall n \geq N_2, |b_n - b| < \varepsilon/2 \end{array}$$

Take $N = \max\{N_1, N_2\}$ & use ①

$$\therefore |(a_n + b_n) - (a + b)| < \varepsilon \quad \forall n \geq N$$

$$ii) \quad |\lambda a_n - \lambda a| = \lambda |a_n - a| < \varepsilon$$

$< \varepsilon/\lambda$ for large enough n

$$\begin{aligned} iii) \quad |a_n b_n - ab| &= |a_n b_n - a_n b + a_n b - ab| \\ &\leq a_n |b_n - b| + b |a_n - a| \\ &\leq \underbrace{|a - \varepsilon| \varepsilon + b \varepsilon}_{\text{arbitrarily small (can be made)}} \text{ for large enough } n \end{aligned}$$

iv)

HW

- Extended Defⁿ of Limits

We say a sequence "diverges" if it doesn't converge. However, not all divergences are the same!

$$\{a_n\} \subseteq \mathbb{R}. \quad \text{If } \forall M \in \mathbb{R}, \exists N \in \mathbb{N} \mid \forall n \geq N, n \in \mathbb{N}, a_n > M$$

... then we say $\{a_n\}$ diverges to $+\infty$, written as: $a_n \rightarrow +\infty$.

(similar for $a_n < m, a_n \rightarrow -\infty$)

extended defⁿ of limits!

From now on, to indicate a finite limit, we may write something like $a_n \rightarrow a \in \mathbb{R}$.
 ↪ always look @ context as well.

Conventions:

$$\begin{array}{ll} \infty + a = \infty & \parallel \quad \infty \cdot a = \infty \\ -\infty + a = -\infty & \parallel \quad -\infty \cdot a = -\infty \end{array} \quad \left. \vphantom{\begin{array}{l} \infty + a = \infty \\ -\infty + a = -\infty \end{array}} \right\} a \neq 0$$

$$\begin{array}{ll} \infty + \infty = \infty & \infty \cdot \infty = \infty \\ -\infty - \infty = -\infty & (-\infty)(-\infty) = \infty \\ \infty - \infty = \text{undefined} & \infty(-\infty) = -\infty \end{array}$$

(for using ∞ 's in arithmetic results)

But! $\pm \infty \cdot 0$ can be 0 or $\pm \infty$ or something else!
 (same for $\pm \infty / \pm \infty$)

- Lemma

$$\{a_n\} \subseteq \mathbb{R}, a_n \rightarrow a; \quad \{b_n\} \subseteq \mathbb{R}, b_n \rightarrow b$$

If $a_n \leq b_n$, then $a \leq b$.

Prove yourself!

• Sandwich Theorem

If $a_n \leq b_n \leq c_n$, $a_n \rightarrow \lambda$, $c_n \rightarrow \lambda$,
then $b_n \rightarrow \lambda$.

Proof: If we show $\{b_n\}$ converges, then by Lemma,
 $b_n \rightarrow \lambda$.

$\forall n, b_n \in [a_n, c_n]$.

Fix any $\varepsilon > 0$. For "large enough" n : $a_n, c_n \in (\lambda - \varepsilon, \lambda + \varepsilon)$

Now, $[a_n, c_n] \subseteq (\lambda - \varepsilon, \lambda + \varepsilon)$

$\therefore b_n \in (\lambda - \varepsilon, \lambda + \varepsilon)$

$(\Rightarrow) |b_n - \lambda| < \varepsilon$ ■

• Examples: See S. Kumarasam and/or Beutle-Shorkut.

• **Exercise Set 2.4.3.** Use the sandwich lemma to solve the following.

- (1) Let (a_n) be a bounded (real) sequence and (x_n) converge to 0. Then show that $a_n x_n \rightarrow 0$.
- (2) The sequence $\sqrt{n+1} - \sqrt{n} \rightarrow 0$.
- (3) $x_n := \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}} \rightarrow 1$.
- (4) Let $0 < a < b$. The sequence $((a^n + b^n)^{1/n}) \rightarrow b$.

• **Theorem 2.4.4** (Nested Interval Theorem – Standard Version). Let $J_n := [a_n, b_n]$ be intervals in $\mathbb{R}$ such that $J_{n+1} \subseteq J_n$ for all $n \in \mathbb{N}$. Assume further that $b_n - a_n \rightarrow 0$, that is, the sequence of lengths of J_n 's goes to zero. Then there exists a unique c such that $\cap_n J_n = \{c\}$.

Proof. We have already seen (Theorem 1.3.23) that there exists at least one $c \in \cap_n J_n$. Let $c, d \in \cap_n J_n$. Assume that $c \leq d$. Then, since $a_n \leq c \leq d \leq b_n$, we see that $0 \leq d - c \leq b_n - a_n$. Since $b_n - a_n \rightarrow 0$, it follows from the sandwich lemma that $d - c = 0$. □

→ S. Kumarasam

Corollary
of
Sandwich
Theorem.

2023-09-02 //

⑥

(9) True or False: If (x_n) and (y_n) are sequences such that $x_n y_n \rightarrow 0$, then one of the sequences converges to 0.

Ans:

False:

$$x_n \rightarrow 0, 1, 0, 1, 0, 1, \dots$$

$$y_n \rightarrow 1, 0, 1, 0, 1, 0, \dots$$

$$x_n y_n \equiv 0 \quad \therefore x_n y_n \rightarrow 0. \quad \text{But } x_n \not\rightarrow 0, y_n \not\rightarrow 0.$$

⑥

(11) Let $a_n \rightarrow 0$. What can you say about the sequence (a_n^n) ?

Ans:

Fix any $\varepsilon > 0$.

$$\exists N \in \mathbb{N} \mid \forall n \geq N, n \in \mathbb{N} \quad |a_n| < \min\{\varepsilon, 1\}$$

$$\therefore |a_n| < 1, \quad |a_n^n| < |a_n| < \min\{\varepsilon, 1\}$$

$$\therefore a_n^n \rightarrow 0$$

⑥

(14) Let (x_n) be a sequence. Assume that $x_n \rightarrow 0$. Let $\sigma: \mathbb{N} \rightarrow \mathbb{N}$ be a bijection. Define a new sequence $y_n := x_{\sigma(n)}$. Show that $y_n \rightarrow 0$.

Ans:

Fix any $\varepsilon > 0$.

$$\therefore x_n \rightarrow 0. \quad \exists N \in \mathbb{N} \mid \forall n \geq N, n \in \mathbb{N} \mid |x_n| < \varepsilon.$$

Now, $\{\sigma^{-1}(1), \sigma^{-1}(2), \sigma^{-1}(3), \dots, \sigma^{-1}(N-1)\} \subseteq \mathbb{N}$ is finite.

$$\therefore N' := \max\{\sigma^{-1}(1), \dots, \sigma^{-1}(N-1)\} + 1.$$

$$\therefore \forall n \geq N', n \in \mathbb{N}, \quad \sigma(n) \geq N$$

$$\Rightarrow \forall n \geq N', n \in \mathbb{N}, \quad |x_{\sigma(n)}| < \varepsilon$$

$$\therefore y_n := x_{\sigma(n)} \rightarrow 0.$$

① Subsequences

- Defⁿ: Let $\{x_n\} \subseteq \mathbb{R}$ be a sequence (in $\mathbb{R}$)
 $\& \{n_k\} \subseteq \mathbb{N}$ be a strictly increasing
sequence (in $\mathbb{N}$).

Then, $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$.

Basically x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 ...
1 2 3 4 5 6 7 8 ...
 $\hookrightarrow$ Subsequence: x_2 x_4 x_6 x_8 ...

- Result: $x_n \rightarrow x \iff \forall \{x_{n_k}\} \subseteq_{\text{seq}} \{x_n\}$
 $\underbrace{x_n \rightarrow x}_P \iff \underbrace{x_{n_k} \rightarrow x}_Q$

Proof:

$P \Rightarrow Q$ (easy)

$P \Rightarrow Q$

- If $x_n \rightarrow x$, then trivial, etc.
- If $x_{n_k} \not\rightarrow x$, then we have to show $\exists$ at least one subsequence $\rightarrow x$.

$\therefore x_n \not\rightarrow x, \forall \varepsilon > 0$, there are infinitely many $n \mid |x_n - x| \geq \varepsilon$.

Turn those infinitely many n into a sequence

HW: Fill in the details.

- Monotone Subsequence Theorem

Every real sequence $\{x_n\} \subseteq \mathbb{R}$ has a monotone subsequence.

Proof: Consider the set:

$$S = \{n \in \mathbb{N} \mid x_m < x_n \ \forall m > n\}$$

Case-I: S is infinite.

Write $S = \{n_1, n_2, n_3, \dots\}$ where $n_1 < n_2 < \dots$

By construction of S : $x_{n_1} > x_{n_2} > x_{n_3} > \dots$

Hence, we have a monotone decreasing subseq. of $\{x_n\}$ ■

Case-II: S is finite

Can get $n_1 > \max S$.

$$\Rightarrow \exists n_2 \in \mathbb{N}, n_2 > n_1 \mid x_{n_2} \geq x_{n_1}$$

Crucially, $n_2 > n_1 > \max S$.

$$\Rightarrow \exists n_3 \in \mathbb{N}, n_3 > n_2 \mid x_{n_3} \geq x_{n_2}$$

$$\therefore \text{Can get } n_1 < n_2 < n_3 < \dots \mid \\ x_{n_1} \leq x_{n_2} \leq x_{n_3} \leq \dots$$

Hence, we have a monotonically increasing subsequence of $\{x_n\}$ ■.

• Bolzano-Weierstrass Theorem

Every bounded sequence has a convergent subsequence.

Proof (for $\mathbb{R}$): Consider $\{x_n\} \in \mathbb{R}$, bounded.

... has a monotone subseq $\{x_{n_k}\}$ which is also bounded
 $\Rightarrow$ convergent.

NOTE: While the proof for $\mathbb{R}$ is easy, Bolzano-Weierstrass holds for a general metric space as well. Then, the proof is much more difficult (see Bartle-Shurman)

• Standard Limits

Theorem 2.5.1. We have the following important limits:

(1) Let $0 \leq r < 1$ and $x_n := r^n$. Then $x_n \rightarrow 0$.

(2) Let $-1 < t < 1$. Then $t^n \rightarrow 0$.

(3) Let $|r| < 1$. Then $nr^n \rightarrow 0$.

(4) Let $a > 0$. Then $a^{1/n} \rightarrow 1$.

(5) $n^{1/n} \rightarrow 1$.

(6) Fix $a \in \mathbb{R}$. Then $\frac{a^n}{n!} \rightarrow 0$.

Result: Convergent $\Rightarrow$ Bounded

HW

Theorem Suppose $x_n \rightarrow x$.

Define $S_n := \frac{x_1 + x_2 + \dots + x_n}{n}$. Then $S_n \rightarrow x$

Essentially: convergence $\Rightarrow$ Cesàro convg. Cesàro convergence

WARNING Converse doesn't hold. Counterexample e.g.

Proof: w.l.o.g assume $x=0$ HW: Justify

$\therefore x_n \rightarrow 0. \exists N \in \mathbb{N} \mid \forall n \geq N, n \in \mathbb{N} \mid |x_n| < \frac{\epsilon}{2}$

$\therefore x_n$ is convergent, it is bounded $\Rightarrow |x_n| < M \forall n$

$$\exists N' \mid n \geq N' \Rightarrow \frac{MN}{n} < \frac{\epsilon}{2}$$

Observe, for $n \geq \max\{N, N'\}$

$$|S_n| = \frac{|x_1 + \dots + x_n|}{n} = \frac{|(x_1 + \dots + x_N) + (x_{N+1} + \dots + x_n)|}{n}$$

$$\leq \frac{MN}{n} + \left(\frac{n-N}{n}\right) \frac{\epsilon}{2} \leq \epsilon$$

Lemma If $\{x_n\}$ is divergent it is either --

i) unbounded, or --

ii) has 2 subsequences converging to different limits.

↪ equivalent to the above Lemma.

- Theorem A bounded sequence $\{x_n\}$ is convergent iff every convergent subsequence converges to the same limit. ↪ also the limit of $\{x_n\}$.
- Theorem A sequence $\{x_n\}$ is convergent iff every subsequence $\{x_{n_k}\}$ has a further subsequence $\{x_{n_{k_l}}\}$ s.t. $\{x_{n_{k_l}}\}$ converges to the same limit.

$$x_n \rightarrow x \iff \forall \{x_{n_k}\} \subseteq_{\text{seq}} \{x_n\}, \exists \{x_{n_{k_l}}\} \subseteq_{\text{seq}} \{x_{n_k}\} \\ x_{n_{k_l}} \rightarrow x \text{ as } l \rightarrow \infty$$

= (2023-09-03)

○ Cauchy Criterion & Completeness

↪ metric space (assume $\mathbb{R}$)

- Defⁿ: A sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq X$ is called Cauchy iff $\forall \varepsilon > 0, \exists N \in \mathbb{N} \mid$

$$\forall m, n \in \mathbb{N}, m > n > N \quad d(x_m, x_n) < \varepsilon$$

↪ metric: $|x_m - x_n|$

- Basically, the terms of the sequence get bunched up as we go. Essentially, the sequence "wants" to converge.
- Theorem $\{x_n\}$ convergent $\Rightarrow \{x_n\}$ Cauchy

Trivial using Δ inequality.

→ only complete metric space.

- Theorem In $\mathbb{R}$, $\{x_n\}$ Cauchy $\Rightarrow \{x_n\}$ convergent.

(won't prove)

- Observe, in $\mathbb{Q}$, $\{x_n\}$ Cauchy $\nRightarrow \{x_n\}$ convergent.

$$\text{Let } S_1 = 1. \quad S_{n+1} = \frac{1}{2} \left(S_n + \frac{2}{S_n} \right).$$

Can show, $S_n \uparrow \sqrt{2}$ in $\mathbb{R}$.

Newton-Raphson

(x2)

$\therefore \{S_n\}$ is Cauchy. Also $\forall n \in \mathbb{N}$, $S_n \in \mathbb{Q}$.
Hence, $\{S_n\}$ is Cauchy in $\mathbb{Q}$. But, $\sqrt{2} \notin \mathbb{Q}$.

So, $\{S_n\}$ can't converge... (in $\mathbb{Q}$)

We say, $\mathbb{Q}$ is not "complete" while $\mathbb{R}$ is.

- Defⁿ: A metric space X is said to be complete iff
in X , Cauchy $\Rightarrow$ convergent.

In practice, this means X has no "holes" that a Cauchy sequence can fall into, instead of converging.

- Result: $\forall x \in \mathbb{R}$, $\exists \{r_n\} \subseteq \mathbb{Q}$ & $\exists \{s_n\} \subseteq \mathbb{Q}^c$ |
 $r_n \rightarrow x$ & $s_n \rightarrow x$.

Basically, every real number has a rational & irrational sequence leading to it.

Proof: $r_n := \frac{1}{10^n} \lfloor 10^n x \rfloor$, $s_n := r_n + 2^{-(\frac{2n+1}{2})}$
works.

⑥ Subsequences (contd)

- Defⁿ: If a bounded seq. $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\} \mid x_{n_k} \rightarrow x^*$, then x^* is called a subsequential limit of $\{x_n\}$.
- If all subsequential limits agree, then the sequence is convergent to that limit (Already done)

$\left(1 + \frac{1}{n}\right)^n \uparrow e \in \mathbb{R} \setminus \mathbb{Q} \quad e \approx 2.718281\dots$
 $\uparrow$: converges (monotonically increasing)
 $\downarrow$: " (" decreasing)

⑥ Limit superior & limit inferior

(Refer Bartle-Sherbert Pg 82)

- Consider $a_n := (-1)^n + \frac{2}{n}$. The subsequential limits are ± 1 .
 But, the largest element is $a_2 = 2$.
 The idea is to capture the interval where the sequence remains active.

- Defⁿ:

$$\limsup_{n \rightarrow \infty} x_n := \lim_{n \rightarrow \infty} \sup \{x_m \mid m \geq n, m \in \mathbb{N}\}$$

$$\liminf_{n \rightarrow \infty} x_n := \lim_{n \rightarrow \infty} \inf \{x_m \mid m \geq n, m \in \mathbb{N}\}$$

- Result: If x^* is a subsequential limit of $\{x_n\}$, then

$$\liminf \{x_n\} \leq x^* \leq \limsup \{x_n\}$$

In fact, if S is the set of all subsequential limits,

$$\liminf \{x_n\} = \inf S; \quad \limsup \{x_n\} = \sup S$$

- Corollary $\{x_n\}$ is convergent iff $\liminf = \limsup$
(= $\lim$)

- While limit may or may not exist, if we allow $\pm\infty$, $\limsup$ & $\liminf$ of a sequence always exists.

= (2023-09-05)

Doubt:

~ Aarjav

3.4.2 Theorem If a sequence $X = (x_n)$ of real numbers converges to a real number x , then any subsequence $X' = (x_{n_k})$ of X also converges to x .

3.4.9 Theorem Let $X = (x_n)$ be a bounded sequence of real numbers and let $x \in \mathbb{R}$ have the property that every convergent subsequence of X converges to x . Then the sequence X converges to x .

why not every?

P: all subsequential limits are same
(i.e. all convergent subsequences converge to some limit)

Q: main sequence converges

R: all subsequences converge (to the same limit)

$$\underbrace{P \Rightarrow Q}_{3.4.9} \Rightarrow \underbrace{Q \Rightarrow R}_{3.4.2} \Rightarrow P$$

★ Series

○ Introduction

- Defⁿ: Let $\{a_n\} \in \mathbb{R}$. Denote $S_n := \sum_{k=1}^n a_k$.

Then $\{S_n\}$, the sequence of partial sums is called the series of $\{a_n\}$.

- Eg.: Let $\{a_n := \frac{1}{n}\}_{n \in \mathbb{N}}$ be the sequence.

Then:

$$\begin{aligned} S_1 &= 1 \\ S_2 &= 1 + \frac{1}{2} \\ S_3 &= 1 + \frac{1}{2} + \frac{1}{3} \\ S_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \\ &\vdots \end{aligned}$$

- Observation $\{a_n\} \equiv \{S_n - S_{n-1}\}_{n \in \mathbb{N}}$ with
will assume in general. $\leftarrow$ convention: $S_0 = 0$.

- Observation $\{a_n\}$ non-negative $\Leftrightarrow \{S_n\}$ monotone $\uparrow$

- Defⁿ: If $S_n \rightarrow S$, we say $\sum_{k=1}^{\infty} a_k$ converges,
and $\sum_{k=1}^{\infty} a_k := S$. Also, we say $\{a_n\}$ is summable if
(we are defining infinite sums now!) $S \in \mathbb{R}$.

- Lemma $\{a_n\}$ summable $\Rightarrow a_n \rightarrow 0$.

Proof: $a_n = S_n - S_{n-1}$

$$\Rightarrow \lim_{n \rightarrow \infty} a_n = S - S = 0$$

- Lemma (Series Arithmetic)

$$\begin{array}{l} \{a_n\} \rightsquigarrow \{A_n\} \text{ sequence of partial sums } \rightarrow A \\ \{b_n\} \rightsquigarrow \{B_n\} \text{ " " " " } \rightarrow B \end{array}$$

i) $a_n \leq b_n \Rightarrow A_n \leq B_n \Rightarrow A \leq B$.

ii) $\{a_n + \lambda b_n\} \rightsquigarrow \{A_n + \lambda B_n\} \rightarrow A + \lambda B$

$\therefore$ Order preserving on $[-\infty, \infty]$ & linear.

- Result: $\{a_n\}$ non-negative then
 $\{S_n\}$ bounded $\Leftrightarrow \{a_n\}$ summable.

Proof: Use MCT. (Monotone Convergence Theorem)

- Result (Cauchy Criterion for Series)

$$\{a_n\} \text{ summable} \Leftrightarrow \forall \varepsilon > 0 \exists N \in \mathbb{N} \mid$$

$$\forall m > n \geq N \quad \underbrace{\left| \sum_{k=n+1}^m a_k \right|}_{S_m - S_n} < \varepsilon$$

- Defⁿ: $\sum_{k=1}^{\infty} a_k$ is said to be absolutely convergent iff $\sum_{k=1}^{\infty} |a_k|$ is convergent.
 $\Leftrightarrow \{ |a_k| \}$ is summable. not commonly used.

We may call $\{a_n\}$ "absolutely summable".

If $\sum_{k=1}^{\infty} a_k$ is convergent but not absolutely convergent, then we say $\sum_{k=1}^{\infty} a_k$ is conditionally convergent.
 Call $\{a_n\}$ "conditionally summable".

If $\{a_n\}$ non-negative, summable $\Leftrightarrow$ absolutely summable.

Examples:

i) $\sum_{j=1}^{\infty} 2^{-j} = 1$ (absolutely convergent)

ii) $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ (divergent)

In fact $\{p_n\}$ be sequence of partial sums, then $p_n \sim \Theta(\log(n))$

discussion in Example 3.3.3(b)) and also for the variety of proofs of its divergence. Here is a proof by contradiction. If we assume the series converges to the number S , then we have

$$\begin{aligned} S &= \left(1 + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6}\right) + \cdots + \left(\frac{1}{2n-1} + \frac{1}{2n}\right) + \cdots \\ &> \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{6} + \frac{1}{6}\right) + \cdots + \left(\frac{1}{2n} + \frac{1}{2n}\right) + \cdots \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots \\ &= S. \end{aligned}$$

The contradiction $S > S$ shows the assumption of convergence must be false and the harmonic series must diverge.

See Bartle-Sharbot for further details.

Basel Problem $\rightarrow$ Watch 3b1b & Mathologer's video.

$$(ii) \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad (\text{absolutely convergent})$$

(c) The 2-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Since the partial sums are monotone, it suffices (why?) to show that some subsequence of (s_k) is bounded. If $k_1 := 2^1 - 1 = 1$, then $s_{k_1} = 1$. If $k_2 := 2^2 - 1 = 3$, then

$$s_{k_2} = \frac{1}{1} + \left(\frac{1}{2^2} + \frac{1}{3^2} \right) < 1 + \frac{2}{2^2} = 1 + \frac{1}{2},$$

and if $k_3 := 2^3 - 1 = 7$, then we have

$$s_{k_3} = s_{k_2} + \left(\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} \right) < s_{k_2} + \frac{4}{4^2} < 1 + \frac{1}{2} + \frac{1}{2^2}.$$

By Mathematical Induction, we find that if $k_j := 2^j - 1$, then

$$0 < s_{k_j} < 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \dots + \left(\frac{1}{2}\right)^{j-1}.$$

Since the term on the right is a partial sum of a geometric series with $r = \frac{1}{2}$, it is dominated by $1/(1 - \frac{1}{2}) = 2$, and Theorem 3.7.5 implies that the 2-series converges.

In fact: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges for $p \in [0, 1]$
converges for $p > 1$.

See Bartle-Shorbert for more details.

$$iv) \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2 \quad (\text{conditionally convergent})$$

• Riemann Rearrangement Theorem

If $\sum a_n$ is conditionally convergent, then:

$$\forall a \in \mathbb{R}, \quad \exists \sigma: \mathbb{N} \rightarrow \mathbb{N} \text{ bijection} \mid \sum a_{\sigma(n)} = a$$

Essentially, you can permute the elements to get any limit!
(order of summation matters!)

Motivation

What is special about a conditionally convergent series?

$\{a_n\} \leftarrow$ conditionally summable seqⁿ.

↪ has both positive & negative terms! ^{include 0}

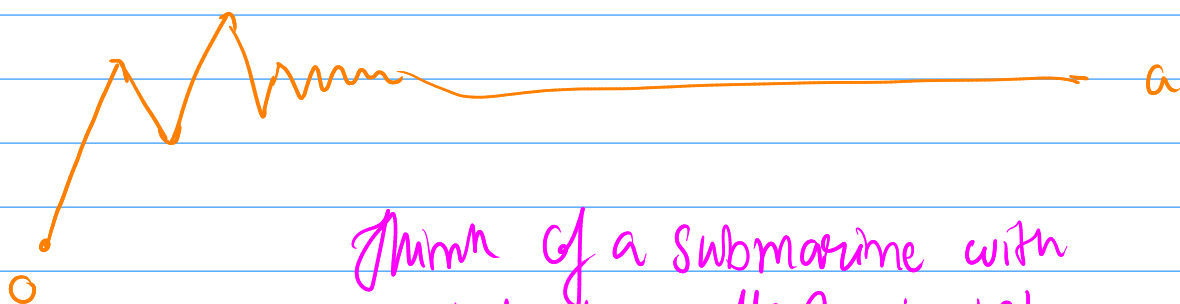
Take a subsequence $\{a_n^+\}$ along all +ve terms
 $\{a_n^-\}$ " " -ve terms.

Observe: $\sum a_n^+ = \infty$, $\sum a_n^- = -\infty$

If not, we would have absolute convergence (both finite)
or divergence (only one finite)

So, we kind of have an $\infty - \infty$ situation.

↪ we can go up or down an arbitrary amount simply by adding appropriate terms to the running sum, and we can do so infinitely often.



Think of a submarine with infinite ballast tanks!

(Proof is based on this, out of scope)

② Product of Series

- Consider $\{a_n\} \in \mathbb{R}$, $\{b_n\} \in \mathbb{R}$.
Let $\{A_n\}$ & $\{B_n\}$ denote the sequence of partial sums respectively.

$$(a_1 + a_2)(b_1 + b_2) = \overset{\text{First}}{a_1 b_1} + \overset{\text{Outer}}{a_1 b_2} + \overset{\text{Inner}}{a_2 b_1} + \overset{\text{Last}}{a_2 b_2} \rightarrow \text{FOIL}$$

$$(a_1 + a_2 + a_3)(b_1 + b_2 + b_3)$$

$$= \begin{array}{c} \begin{array}{ccc} & b_1 & b_2 & b_3 \\ a_1 & a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 & a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 & a_3 b_1 & a_3 b_2 & a_3 b_3 \end{array} \end{array} \left\{ \begin{array}{l} a_1 b_1 + a_1 b_2 + a_1 b_3 + \\ a_2 b_1 + a_2 b_2 + a_2 b_3 + \\ a_3 b_1 + a_3 b_2 + a_3 b_3 \end{array} \right.$$

$$\begin{array}{llll} s_2 := & a_1 b_1 + & \leftarrow & \text{Sum of indices} \\ s_3 := & a_1 b_2 + a_2 b_1 + & \leftarrow & 2 \\ s_4 := & a_1 b_3 + a_2 b_2 + a_3 b_1 + & \leftarrow & 3 \\ s_5 := & a_2 b_3 + a_3 b_2 + & \leftarrow & 4 \\ s_6 := & a_3 b_3 & \leftarrow & 5 \end{array}$$

based on diagonals

Convention for this Section: We will use $\mathbb{N}_0$ instead of $\mathbb{N}$, for indexing.
↳ Prevents "shifting" $1 \rightarrow 2$ when multiplying.

Otherwise index set of $\{s_n\}$ changes!

NOTE: A_n is now $\sum_{k=0}^{n-1} a_k$ (still n elements)
 $a_0, \dots, a_{n-1}$

- Observe: Using this diagonal trick...

$$A_n B_n = \sum_{k=0}^{2n-2} c_k \quad \text{where} \quad c_k = \sum_{\ell=0}^k a_\ell b_{k-\ell}$$

→ Set $n=4$ & compute c_5 :

$$c_5 = a_0 b_5 + a_1 b_4 + a_2 b_3 + a_3 b_2 + a_4 b_1 + a_5 b_0$$

→ not present in $A_4 B_4$ →

∴ Need to redefine: $c_k = \sum_{\ell=0}^k \bar{a}_\ell \bar{b}_{k-\ell}$

$$\text{where } \bar{a}_\ell := a_\ell \mathbb{1}_{\{0 \leq \ell < n\}} \\ \bar{b}_\ell := b_\ell \mathbb{1}_{\{0 \leq \ell < n\}}$$

- Now, $\sum_{k=0}^{2n-2} c_k$ is a partial sum.

∴ As $n \rightarrow \infty$, we have the series $\sum_{k=0}^{\infty} c_k$.

$$\text{Now, } A_n \rightarrow A, B_n \rightarrow B, \therefore \sum_{k=0}^{\infty} c_k = AB$$

Not entirely true!

Technicality: c_n 's depend on n !!

Taking $n \rightarrow \infty$ is not a trivial step...

- Mertens' Theorem

If $\{a_n\}, \{b_n\}$ summable, and
 at least one absolutely summable, then: R solves

$$\left(\sum_{n=0}^{\infty} a_n \right) \left(\sum_{n=0}^{\infty} b_n \right) = \sum_{n=0}^{\infty} c_n, \quad \text{where } c_k = \sum_{\ell=0}^k a_\ell b_{k-\ell}$$

series → Cauchy product.

- To see why we need absolute convergence---

i) Take the case of $\{a_n\}, \{b_n\}$ non-negative, summable
 $\Rightarrow \{A_n\}, \{B_n\}$ monotone bounded
 $\Rightarrow \{A_n B_n\}$ monotone bounded
 $\Rightarrow \{A_n B_n\}$ convergent $\Rightarrow$ Cauchy product converges.

We may see this as:

NOT ♥ Even when a_n 's change with n , they just add some stuff, boundedness of $\{A_n B_n\}$ saves us.

ii) Contrast w/ $\{a_n\}, \{b_n\}$ conditionally summable
 (eg: A.M.'s: alternating harmonic series)

The changes in a_n destroy the "niceness" of our Cauchy product. $\hookrightarrow$ (can be +ve or -ve...)

The actual proof is out of scope.

- Application Factoring of infinite series:
 Especially where you have a summation like in ex.

- Result (weaker version of Mertens' Theorem)

$\{a_n\}, \{b_n\}$ Cesàro summable $\Rightarrow$ The Cauchy product (series) is Cesàro convergent.
 $\Updownarrow$
 $\{A_n\}, \{B_n\}$ Cesàro convergent.

$\hookrightarrow \frac{A_1 + A_2 + \dots + A_n}{n} \rightarrow A \Leftrightarrow \sum_{i=0}^n \left(\frac{n-i}{n}\right) a_i \rightarrow A \text{ as } n \rightarrow \infty$
 $\rightarrow$ pre-factor.

- Connection with Convolutions

Suppose you have $f, g: \mathbb{N}_0 \rightarrow \mathbb{C}$.

Then: $(f * g)(n) = \sum_{i+j=n} f(i) g(j)$

If, we defined sequences $\{a_n\}, \{b_n\}$ using f, g resp.

then, $(f * g)(n) = \sum_{i=0}^n a_i b_{n-i} = \underline{\underline{c_n}}!!$

Thus, convolution is identical to the Cauchy product.

👁
Note to
the reader

Motivating convolutions is a beautiful topic I'll leave you to explore yourself. I'll send certain videos on it, but there is much, much more.

(Eg: How Fourier transforms turn convolutions into multiplications.)

See Also: Polynomial multiplication

↳ leads to Cauchy product of power series.

↓ watch ↓

<https://www.youtube.com/watch?v=KuXjwB4LzSA>

In general, follow 3blue1brown.

♥ Essence of calculus

♥ Essence of linear algebra (after exams)

① More on Cesàro

$$\begin{aligned} & \overbrace{1-1+1-1+1-1+\dots}^{-S} =: S \\ & \Rightarrow 1-S = S \\ & \Rightarrow S = 1/2 \quad (!?) \end{aligned}$$

- Let us consider the sequence:

$$+1, -1, +1, -1, +1, -1, +1, -1, \dots$$

Clearly, this doesn't converge.

Nonetheless, define a_n, A_n appropriately, and consider $A_n = \frac{A_1 + \dots + A_n}{n}$.

$$\text{Then: } A_n = \begin{cases} 1 & , n=1, 3, 5, \dots \text{ (odd)} \\ 0 & , n=2, 4, 6, \dots \text{ (even)} \end{cases}$$

$$\therefore A_n = \frac{1}{n} \left\lfloor \frac{n+1}{2} \right\rfloor \rightarrow \frac{1}{2} \text{ as } n \rightarrow \infty$$

This is because the sequence $(1, 0, 1, 0, 1, 0, \dots) \rightarrow 1/2$ Cesàro.

$\therefore (+1, -1, +1, -1, \dots)$ is Cesàro summable, with sum $1/2$.

- Let us consider a permutation of the sequence...

$$(+1, +1, -1, +1, +1, -1, +1, +1, -1, \dots)$$

$\uparrow$
 b_1

$$\therefore B_n = n - 2 \left\lfloor \frac{n}{3} \right\rfloor$$

$$\begin{aligned} \therefore B_n &= \frac{B_1 + \dots + B_n}{n} \approx \frac{\frac{n(n+1)}{2} - 2 \cdot 3 \cdot \frac{\left\lfloor \frac{n}{3} \right\rfloor \left(\left\lfloor \frac{n}{3} \right\rfloor + 1 \right)}{2}}{n} \\ &\rightarrow \infty \text{ as } n \rightarrow \infty \end{aligned}$$

$\therefore$ Cesàro summations change upon permutations.

- Consider the sequence:
 $(+1, -2, +3, -4, +5, -6, \dots)$

$$c_n = (-1)^{n-1} n$$

$$C_n = (-1)^{n-1} \left\lfloor \frac{n+1}{2} \right\rfloor \rightarrow (1, -1, 2, -3, 3, -3, \dots)$$

$$G_n = \frac{c_1 + \dots + c_n}{n} = \begin{cases} 0 & n \text{ is even} \\ \frac{1}{n} \left\lfloor \frac{n+1}{2} \right\rfloor & n \text{ is odd.} \end{cases}$$

Observe G_n doesn't converge. But we are already in Césaro-land!

$$\mathbb{G}_n = \frac{G_1 + \dots + G_n}{n} \rightarrow \left(\frac{1}{4} \right)$$

$$S_2 = 1 - 2 + 3 - 4 + 5 - 6 + \dots$$

$$S_2 = 1 - 2 + 3 - 4 + 5 - \dots$$

$$2S_2 = 1 - 1 + 1 - 1 + 1 - 1 + \dots = 1/2 \text{ in Césaro}$$

$$\Rightarrow S_2 = 1/n \{ ? \} \text{ can't be a coincidence}$$

- $S = 1 + 2 + 3 + 4 + 5 + \dots \rightarrow$ doesn't converge
 $-S_2 = -1 + 2 - 3 + 4 - 5 + \dots$ at all! ($\uparrow \infty$)

$$S - S_2 = 4S$$

$$\Rightarrow S = -\frac{1}{3} S_2 = -\frac{1}{12}$$

$$\zeta(s) = \sum_{n \in \mathbb{N}} n^{-s}$$

$$\nearrow \text{sg} \textcircled{1} \text{Re}(s) > 1$$

Turns out $-\frac{1}{12}$ is the value of the analytic continuation of the Riemann zeta $f^n @ -1 \dots$

Moral: Don't mess w/ Césaro.

⑤ Tests for Convergence

• Comparison Test

Consider $0 \leq \{a_n\} \leq \{b_n\}$, then:

$$\{b_n\} \text{ summable} \Rightarrow \{a_n\} \text{ summable}$$

Proof: $\{b_n\}$ summable

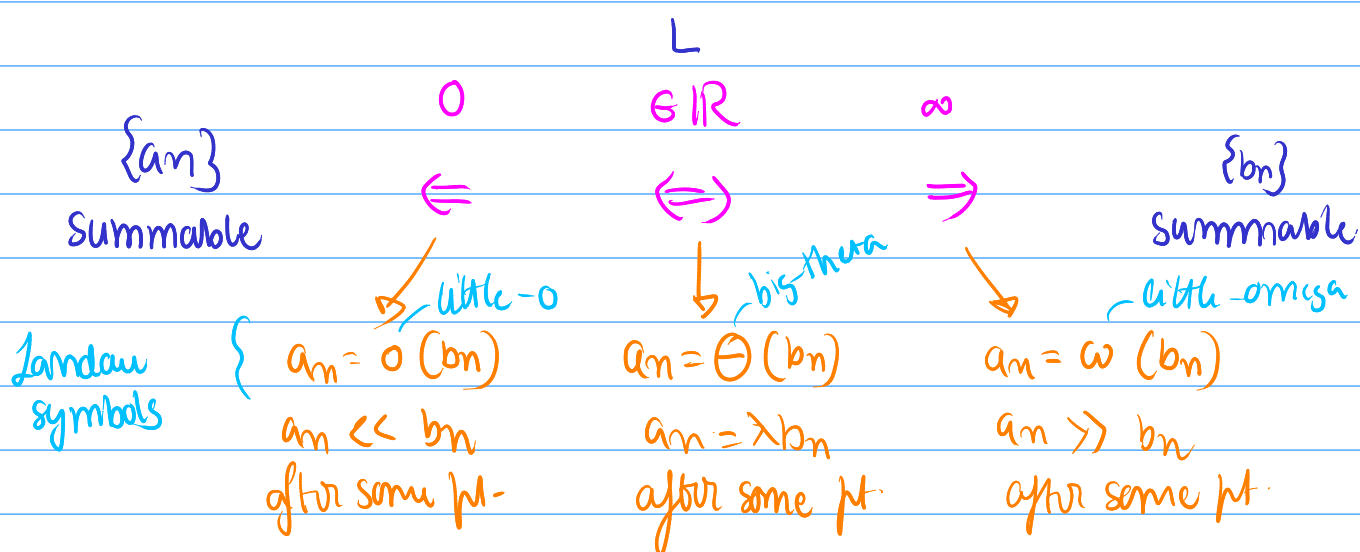
$$\Leftrightarrow \{b_n\} \text{ convergent} \Rightarrow \{b_n\} \text{ bounded.}$$

$\therefore 0 \leq \{a_n\} \leq \{b_n\}$, then $\{a_n\}$ bounded, monotone $\uparrow$.

Hence, $\{a_n\}$ convergent $\Rightarrow \{a_n\}$ summable.

• Limit Comparison Test (LCT) $\rightarrow$ growth rates comparison.

Consider $0 < \{a_n\}, \{b_n\}$. Denote $L = \lim_{n \rightarrow \infty} \frac{a_n}{b_n}$



⑥ Prove that $\sum_{n \in \mathbb{N}} \frac{1}{n^2 - n + 1}$ converges.

DIY

limit: Use LCT with $\frac{1}{n^2}$ ---

$$\frac{\frac{1}{n^2 - n + 1}}{\frac{1}{n^2}} = \frac{1}{1 - (\frac{1}{n}) + (\frac{1}{n^2})} \rightarrow 1$$

• Cauchy Condensation Test.

Given $\{a_n\} \geq 0$ monotonically decreasing.

$$\sum_{n \in \mathbb{N}} a_n \text{ convergent} \Leftrightarrow \sum_{n \in \mathbb{N}} 2^n a_{2^n} \text{ convergent}$$

Estimate [edit]

The Cauchy condensation test follows from the stronger estimate,

$$\sum_{n=1}^{\infty} f(n) \leq \sum_{n=0}^{\infty} 2^n f(2^n) \leq 2 \sum_{n=1}^{\infty} f(n),$$

which should be understood as an [inequality of extended real numbers](#). The essential thrust of a [proof](#) follows, patterned after [Oresme's](#) proof of the [divergence](#) of the [harmonic series](#).

To see the first inequality, the terms of the original series are rebracketed into runs whose lengths are [powers of two](#), and then each run is bounded above by replacing each term by the largest term in that run. That term is always the first one, since by assumption the terms are non-increasing.

$$\begin{aligned} \sum_{n=1}^{\infty} f(n) &= f(1) + f(2) + f(3) + f(4) + f(5) + f(6) + f(7) + \dots \\ &= f(1) + (f(2) + f(3)) + (f(4) + f(5) + f(6) + f(7)) + \dots \\ &\leq f(1) + (f(2) + f(2)) + (f(4) + f(4) + f(4) + f(4)) + \dots \\ &= f(1) + 2f(2) + 4f(4) + \dots = \sum_{n=0}^{\infty} 2^n f(2^n) \end{aligned}$$

To see the second inequality, these two series are again rebracketed into runs of power of two length, but "offset" as shown below, so that the run of $2 \sum_{n=1}^{\infty} f(n)$ which begins with $f(2^n)$ lines up with the end of the run of $\sum_{n=0}^{\infty} 2^n f(2^n)$ which ends with $f(2^n)$, so that the former stays always "ahead" of the latter.

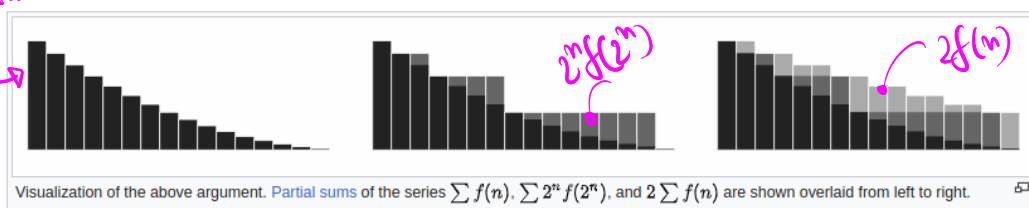
$$\begin{aligned} \sum_{n=0}^{\infty} 2^n f(2^n) &= f(1) + (f(2) + f(2)) + (f(4) + f(4) + f(4) + f(4)) + \dots \\ &= (f(1) + f(2)) + (f(2) + f(4) + f(4) + f(4)) + \dots \\ &\leq (f(1) + f(1)) + (f(2) + f(2) + f(3) + f(3)) + \dots = 2 \sum_{n=1}^{\infty} f(n) \end{aligned}$$

$$f(n) =: a_n$$

$$f(n) =: a_n$$

$$2^n f(2^n)$$

$$2f(n)$$



https://en.wikipedia.org/wiki/Cauchy_condensation_test

Eg.: Consider $\{\frac{1}{n}\}_{n \in \mathbb{N}}$.

$$\begin{aligned}\sum \frac{1}{n} &= 1 + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) + \dots \\ &\geq \underbrace{\frac{1}{2}}_{\frac{1}{2}} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{\frac{1}{2}} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}_{\frac{1}{2}} + \dots \\ &\therefore \sum \frac{1}{n} \text{ diverges.} \quad \left(\frac{1}{2}\right) 2^n a_{2^n}\end{aligned}$$

Eg.: Consider $\{\frac{1}{n^2}\}$

$$\begin{aligned}\sum \frac{1}{n^2} &= \frac{1}{1^2} + \left(\frac{1}{2^2} + \frac{1}{3^2}\right) + \left(\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2}\right) + \dots \\ &\leq \underbrace{\frac{1}{1^2}}_{1 \times \frac{1}{1^2}} + \underbrace{\frac{1}{2^2} + \frac{1}{2^2}}_{2 \times \frac{1}{2^2}} + \underbrace{\frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^2}}_{4 \times \frac{1}{4^2}} + \dots \\ &= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 2 < \infty.\end{aligned}$$

$\therefore \sum \frac{1}{n^2}$ converges.

HW Use this to prove $\sum_{n \in \mathbb{N}} \frac{1}{n^p}$, $p \in [0, \infty)$

converges iff $p > 1$.

Asymptotic Notations / Lambda Symbols

• $f(n) = O(g(n))$ big-o ↗ after some n , $f(n) \leq \text{multiple of } g(n)$
↪ not functions!

$\Leftrightarrow \exists c \in \mathbb{R}^+, N \in \mathbb{N} \mid \forall n \geq N, f(n) \leq cg(n)$

Since we can make c arbitrarily large, $N=1$ works.

• $f(n) = \Omega(g(n)) \Leftrightarrow g(n) = O(f(n))$ big-omega

$\Leftrightarrow \exists c \in \mathbb{R}^+, N \in \mathbb{N} \mid \forall n \geq N, f(n) \geq cg(n)$

↪ ... lower bound multiple of $g(n)$

• $f(n) = \Theta(g(n)) \Leftrightarrow f(n) = O(g(n)) \& f(n) = \Omega(g(n))$ big-theta

$f(n)$ is eventually bounded between two $\&$

multiples of $g(n)$. || Observe: $f(n) = \Theta(g(n)) \Leftrightarrow g(n) = \Theta(f(n))$

• Result: $f(n) = O(g(n)) \Leftrightarrow \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} < \infty$

Corollary: $f(n) = \Theta(g(n)) \Leftrightarrow \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} \in (0, \infty)$

• $f(n) = o(g(n)) \Leftrightarrow \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0$ little-o
(implies big-o)

$f(n) = \omega(g(n)) \Leftrightarrow \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \infty$ little-omega
(implies Ω)

$f(n) \sim g(n) \Leftrightarrow \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 1$ asymptotically equal to.
(implies Θ)

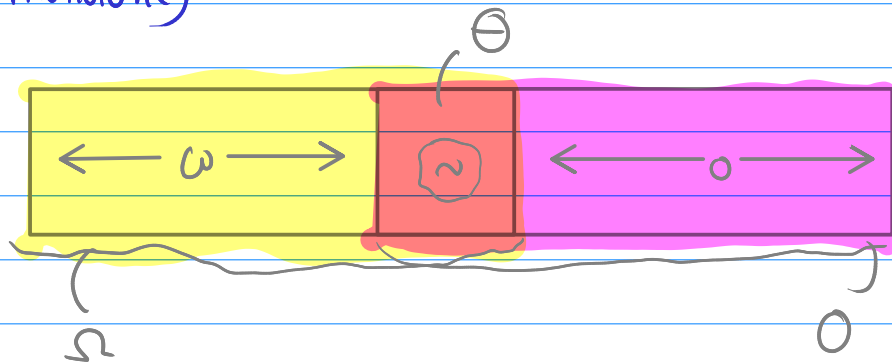
O, Ω inverses; o, ω inverses.

• Examples:

		Strictest
i)	$n^2 = O(n^3)$	O
ii)	$n^2 = O(2n^2 + n)$	Θ
iii)	$n^2 = O(n^2)$	$\sim (=)$
iv)	$n^2 = O(0.1n^2 + 1)$	Θ

TRUE

- Based on this, can classify any 2 pairs of "nice" functions.
(say monotone)



Here,

$$\begin{aligned} \sim &\subseteq \Theta & \Theta &= \bigcirc \cap \bigcirc \\ \bigcirc &= \bigcirc \cup \Theta \\ \bigcirc &= \bigcirc \cup \Theta \end{aligned}$$

⑥ i) $n^2 + 2n + 3 + \frac{1}{n} = \Theta(n^2)$ TRUE

ii) $\frac{1}{n \log n} = \Omega(\frac{1}{n})$ FALSE

iii) $\log(\log(n!)) = O(2^n)$ TRUE

iv) $n^n = 2^{n \log n} = O(2^n)$ FALSE

v) $n^n = O(2^{2^n})$ TRUE

vi) $n^n = \Omega(2^{2^n})$ FALSE

Limit: $n! \sim \frac{1}{\sqrt{2\pi n}} \left(\frac{n}{e}\right)^n$ Stirling's Approximation

• Cauchy Ratio Test

Consider $\{a_n\}$. $L := \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$

If $L > 1$, $\sum a_n$ diverges.
 $L < 1$, $\sum a_n$ converges absolutely.
 $L = 1$, can't say

↳ ex: $\sum \frac{1}{n}$ diverges, $\sum \frac{1}{n^2}$ converges

Stronger version: $R := \limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$

Leaves, $\therefore$ lim may not exist!

$r := \liminf_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$

If $r > 1$, diverges.
 $R < 1$, converges absolutely.
 otherwise, inconclusive.

(Proof is LCT w/ geometric series)

• Defn: A sequence $\{a_n\}$ is called contractive iff $\exists C \in (0, 1)$
 $\& \quad \forall n \in \mathbb{N}, n \geq N$

$$|a_{n+1} - a_n| \leq C |a_n - a_{n-1}|$$

↳ contraction factor

• Result: Contractive $\Rightarrow$ Cauchy $\xrightarrow{R}$ Convergent.

• Cauchy Ratio Test: Contractive in $\mathbb{R} \Rightarrow$ Summable.
 $L \rightarrow 0$
 my FDNLS

• Cauchy Root Test

Consider $\{a_n\} \subseteq \mathbb{R}$. $L := \limsup_{n \rightarrow \infty} |a_n|^{1/n}$

If $L > 1$, diverges.

$L < 1$, converges absolutely,

$L = 1^+$ ($L = 1$, limit is from above) diverges,
otherwise inconclusive. $\leadsto \sum \frac{1}{n}, \sum \frac{1}{n^2} \dots$

3.36 Remarks The ratio test is frequently easier to apply than the root test, since it is usually easier to compute ratios than n th roots. However, the root test has wider scope. More precisely: Whenever the ratio test shows convergence, the root test does too; whenever the root test is inconclusive, the ratio test is too. This is a consequence of Theorem 3.37, and is illustrated by the above examples.

Neither of the two tests is subtle with regard to divergence. Both deduce divergence from the fact that a_n does not tend to zero as $n \rightarrow \infty$.

//

• Solve questions from TOMATO.

Use Bartle-Shenitzer & S. Kumaresan as primary refs.

// END.